Decomposition of the covariant derivative of the curvature tensor of a pseudo-Kählerian manifold

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Abstract

The decompositions of the curvature tensors of pseudo-Riemannian and pseudo-Kählerian manifolds are reviewed. The known decomposition of the covariant derivative of the curvature tensor of a pseudo-Riemannian manifold is refined. A decomposition of the covariant derivative of the curvature tensor of a pseudo-Kählerian manifolds is obtained.

1 Introduction

It is well known that the curvature tensor R of any pseudo-Riemannian manifold (M,g) can be decomposed into the following three components:

the Weyl tensor W, which is the totally trace-free part of R;

the trace-free part Ric^0 of the Ricci tensor Ric of (M,g);

the scalar curvature s.

This decomposition is the consequence of the fact that the space $\mathcal{R}(\mathfrak{so}(p,q))$ of algebraic curvature tensors (i.e. the space of possible values of the curvature tensor of a pseudo-Riemannian manifold (M,g)) decomposes into the direct sum of three irreducible $\mathfrak{so}(p,q)$-modules if $p+q\geq 5$, where (p,q) is the signature of g. Setting to zero some of these components gives rise to different geometrical characterizations of (M,g):

$$\begin{aligned} W=0 \quad &\Leftrightarrow \quad (M,g) \text{ is conformally flat;}\\ \mathrm{Ric}^0=0 \quad &\Leftrightarrow \quad (M,g) \text{ is an Einstein manifold;}\\ \mathrm{Ric}^0=0 \text{ and } s=0 \quad &\Leftrightarrow \quad (M,g) \text{ is Ricci-flat;}\\ W=0 \text{ and } \mathrm{Ric}^0=0 \quad &\Leftrightarrow \quad (M,g) \text{ has constant section curvature.}\end{aligned}$$

In [9] Gray showed that the space $\mathcal{R}ic^\nabla$ of possible values of the covariant derivative of the Ricci tensor of (M,g) decomposes into the direct sum of three irreducible $\mathfrak{so}(p,q)$-modules. Setting to zero some of the corresponding components of the covariant derivative of the Ricci tensor gives 6 equations, that are generalizations of the Einstein equation. Chapter 16 from [3] is dedicated to these equations.

In [16] Strichartz decomposed the space $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ of covariant derivatives of algebraic curvature tensors (i.e. the space of possible values of the covariant derivative of the curvature tensor of a pseudo-Riemannian manifold (M,g)) into four components. He used the $\mathfrak{so}(p,q)$-equivariant linear map

$$\mathrm{tr}_{2,4}:\mathcal{R}^\nabla(\mathfrak{so}(p,q))\to\mathcal{R}ic^\nabla,$$

which has the form

$$\nabla R\mapsto\nabla\,\mathrm{Ric}\,.$$

The kernel of this map is an irreducible submodule $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ and $\mathrm{tr}_{2,4}$ restricted to the other three irreducible submodules of $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ is an isomorphism. Then an inverse isomorphism to this one is constructed.

As the first result of this paper, we refine the decomposition from [16]. For this we consider the $\mathfrak{so}(p,q)$-equivariant linear map

$$\mathrm{tr}_{1,5}:\mathcal{R}^\nabla(\mathfrak{so}(p,q))\to\mathcal{P}(\mathfrak{so}(p,q)).$$

Here the space $\mathcal{P}(\mathfrak{h})$ for a subalgebra $\mathfrak{h}\subset\mathfrak{so}(p,q)$ is defined in the following way:

$$\mathcal{P}(\mathfrak{h})=\{P\in\mathrm{Hom}(\mathbb{R}^{p,q},\mathfrak{h})|g(P(X)Y,Z)+g(P(Y)Z,X)+g(P(Z)X,Y)=0,\quad X,Y,Z\in\mathbb{R}^{p,q}\},$$

where g is the pseudo-Euclidean form on $\mathbb{R}^{p,q}$. The results of [8] allow to find spaces $\mathcal{P}(\mathfrak{h})$ for each irreducible holonomy algebra $\mathfrak{h}\subset\mathfrak{so}(p,q)$ of pseudo-Riemannian manifolds. In particular, the $\mathfrak{so}(p,q)$-module $\mathcal{P}(\mathfrak{so}(p,q))$ is the direct sum of two irreducible modules. The kernel of $\mathrm{tr}_{1,5}$ consists of two irreducible submodules of $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ and $\mathrm{tr}_{1,5}$ restricted to the other two irreducible submodules of $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ is an isomorphism. We construct the inverse isomorphism to this one and find the projections of ∇R to the irreducible parts of the kernel of $\mathrm{tr}_{1,5}$.

We show that the covariant derivative ∇R of the curvature tensor R of a pseudo-Riemannian manifold can be decomposed into the following four components:

the totally trace-free part of ∇R, which coincides with the totally trace-free part of ∇W (we denote this component by S_0');

the symmetrization of the tensor $\nabla\left(\mathrm{Ric}-\frac{2}{n+2}sg\right)$ (we denote this component by S_0'');

the Cotton curvature tensor C;

the gradient $\mathrm{grad}\,s$ of the scalar curvature s.

Comparing with the decomposition from [16], the obtained below components have a simpler form. Especially the explicit form for S_0' is found, while in [16] it is written only that S_0' equals ∇R minus the other three components. The fact that S_0' belongs to the space $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ implies the second Bianchi identity for ∇W [7].

Remark that in [16, 18] an unpublished preprint of Gray and Vanhecke is cited, where the decomposition of the space $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ is also found.

Now we may set to zero some of the obtained components of ∇R and this will give 14 different natural equations. The last three components of ∇R are defined exactly by the same tensors

defining the last three components of ∇R and may be obtained taking a certain trace in the decomposition of ∇R. Hence 6 of the 14 equations are considered in Chapter 16 from [3].

Recall the geometric meaning of some of these equations:

$$\begin{aligned}
C = 0 \quad &\Leftrightarrow \quad W \text{ is harmonic;} \\
C = 0 \text{ and } \operatorname{grad} s = 0 \quad &\Leftrightarrow \quad R \text{ is harmonic} \Leftrightarrow W \text{ is harmonic and } s \text{ is constant} \\
&\qquad \Leftrightarrow \operatorname{Ric} \text{ is a Codazzi tensor;} \\
\nabla_X \operatorname{Ric}(X,X) = 0 \quad &\Leftrightarrow \quad S_0'' = 0 \text{ and } \operatorname{grad} s = 0; \\
\nabla_X \left(\operatorname{Ric}(X,X) - \tfrac{2}{n+2} s g(X,X) \right) = 0 \quad &\Leftrightarrow \quad S_0'' = 0.
\end{aligned}$$

In [5, 6] and in some other papers the equation

$$\nabla W = 0$$

is studied. In particular, if (M, g) is a Riemannian manifold, then this equation implies $W = 0$ or $\nabla R = 0$ [5]. The final step in a local classification of pseudo-Riemannian manifolds with $\nabla W = 0$ is done in [6]. In terms of the decomposition of ∇R the equation $W = 0$ is equivalent to the following two equations:

$$S_0' = 0, \qquad C = 0.$$

It would be interesting to describe pseudo-Riemannian manifolds (M, g) that satisfy the stronger condition

$$S_0' = 0.$$

Then we consider the case of pseudo-Kählerian manifolds. It is well known that in this case the curvature tensor R can be decomposed into the following three components:

the Bochner tensor B, which is the totally trace-free part of R;

the trace-free part Ric^0 of the Ricci tensor Ric of (M, g);

the scalar curvature s.

We introduce some notation simplifying the usual formulas for the decomposition of R.

Pseudo-Kählerian manifolds with $B = 0$ are called Bochner-Kähler. Many results and references concerning these manifolds can be found in the fundamental paper of Bryant [4].

Then we show that the covariant derivative ∇R of the curvature tensor of a pseudo-Kählerian manifold can be decomposed into the following three components:

the totally trace-free part of ∇R, which coincides with the totally trace-free part of ∇B (we denote this component by Q_0);

a tensor D, which is an analog of the Cotton curvature tensor;

the gradient $\operatorname{grad} s$ of the scalar curvature s.

To obtain this decomposition we use our technics developed for general pseudo-Riemannian manifolds.

This decomposition allows to consider 6 natural equations on ∇R. The pseudo-Kählerian manifolds with $D=0$ (i.e. with harmonic Bochner tensor) are studied e.g. in [10]. The pseudo-Kählerian manifolds with $\nabla B=0$ (i.e. with $Q_0=D=0$) are studied e.g. in [12, 13].

The fact that Q_0 belongs to the space $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$ implies the second Bianchi identity for ∇B, which is obtained recently in [15].

In [14] the tensor ∇R for Riemannian and Kählerian manifolds is decomposed in two different ways in the sum of two orthogonal components, this gives some inequalities for the norms of ∇R, $\nabla\,\mathrm{Ric}$ and $\operatorname{grad} s$. The results of the present paper applied to Riemannian and Kählerian manifolds show that $|\nabla R|^2$ equals to the squares of the norms of the components of ∇R. This implies different inequalities (that can be obtained omitting the norms of some components), some of these inequalities are already obtained in [14]. If some of the inequalities become an equality on a manifold (M,g), then the omitted components of ∇R must be zero. These results will be considered in a separate paper.

In Sections 2, 3, 4 and 5 we recall some facts and state the results. In the other sections we prove the results.

Below the letters X,Y,Z,U,V denote either elements of $\mathbb{R}^{p,q}$, or of $\mathbb{R}^{2p,2q}$, or vector fields on (M,g). Similarly, $X_1,...,X_n$ is either a basis of $\mathbb{R}^{p,q}$ or a local basis of vector fields on (M,g). In the pseudo-Kählerian case $X_1,...,X_n,X_{n+1}=JX_1,...,X_{2n}=JX_n$ is either a basis of $\mathbb{R}^{2p,2q}$ or a local basis of vector fields on (M,g,J).

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2 Decomposition of the curvature tensor of a pseudo-Riemannian manifold

Denote be g the pseudo-Euclidean metric on $\mathbb{R}^{p,q}$. Using g we may identify the Lie algebra $\mathfrak{so}(p,q)$ with the space of bivectors $\Lambda^2\mathbb{R}^{p,q}$ in such a way that

$$(X\wedge Y)Z=g(X,Z)Y-g(Y,Z)X.$$

For an element $H\in\odot^2\mathbb{R}^{p,q}$ (which can be considered both as a symmetric linear map and as a symmetric bilinear form) denote by $H\wedge g$ the endomorphism of $\wedge^2\mathbb{R}^{p,q}$ defined by

$$(H\wedge g)(X,Y)=HX\wedge Y+X\wedge HY. \tag{1}$$

The same notation can be used for a pseudo-Riemannian manifold (M,g).

The curvature tensor R of any pseudo-Riemannian manifold (M,g) can be decomposed into the sum of three components in the following way:

$$R=W+\frac{1}{n-2}\left(\mathrm{Ric}-\frac{s}{n}g\right)\wedge g+\frac{s}{2n(n-1)}g\wedge g, \tag{2}$$

where Ric and s are the Ricci tensor and the scalar curvature, respectively. The tensor W is called the Weyl conformal curvature tensor.

Decomposition (2) can be rewritten as

$$R=W+L\wedge g, \tag{3}$$

where

$$L = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{s}{2(n-1)}g\right)$$

is the Schouten tensor.

Let us explain the origin of Decomposition (2). Let (p,q) be the signature of the manifold (M,g), $p+q=n$. For any subalgebra $\mathfrak{h} \subset \mathfrak{so}(p,q)$ consider the space of algebraic curvature tensors

$$\mathcal{R}(\mathfrak{h}) = \{R \in \mathrm{Hom}(\Lambda^2\mathbb{R}^{p,q}, \mathfrak{h}) | R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0\}.$$

In [1] the spaces $\mathcal{R}(\mathfrak{h})$ are found for all irreducible Riemannian holonomy algebras $\mathfrak{h} \subset \mathfrak{so}(n)$. This result allows to find also the spaces $\mathcal{R}(\mathfrak{h})$ for all irreducible holonomy algebras $\mathfrak{h} \subset \mathfrak{so}(p,q)$ of pseudo-Riemannian manifolds. In particular, if $n \geq 5$, then the $\mathfrak{so}(p,q)$-module $\mathcal{R}(\mathfrak{so}(p,q))$ admits the following decomposition into irreducible components:

$$\mathcal{R}(\mathfrak{so}(p,q)) = \mathcal{R}_0(\mathfrak{so}(p,q)) \oplus \mathcal{R}'(\mathfrak{so}(p,q)) \oplus \mathcal{R}_1(\mathfrak{so}(p,q)) \simeq V_{2\bar{\pi}_2} \oplus V_{2\pi_1} \oplus \mathbb{R}, \tag{4}$$

where $\bar{\pi}_2 = 2\pi_2$ if $n=5$ and $\bar{\pi}_2 = \pi_2$ if $n \geq 6$; here and in the next section V_Λ denotes the real irreducible representation of $\mathfrak{so}(p,q)$ defined by the complex irreducible representation of $\mathfrak{so}(n,\mathbb{C})$ with the highest weight Λ. For simple Lie algebras we use the notations from [19]. If $x \in M$ and R is the curvature tensor of (M,g), then, clearly, $R_x \in \mathcal{R}(\mathfrak{so}(T_xM)) \simeq \mathcal{R}(\mathfrak{so}(p,q))$. Decomposition (4) shows that R_x can be written as the sum

$$R_x = R_{0x} + R'_x + R_{1x}$$

where R_{0x}, R'_x and R_{1x} belong to the submodules of $\mathcal{R}(\mathfrak{so}(T_x,M))$ isomorphic to $V_{2\bar{\pi}_2}$, $V_{2\pi_1}$ and $\mathbb{R}$, respectively. Note that $V_{2\pi_1} \oplus \mathbb{R} \simeq \odot^2\mathbb{R}^{p,q}$ and the inclusion $\odot^2\mathbb{R}^{p,q} \hookrightarrow \mathcal{R}(\mathfrak{so}(p,q))$ is given by

$$H \in \odot^2\mathbb{R}^{p,q} \quad \mapsto \quad H \wedge g \in \mathcal{R}(\mathfrak{so}(p,q)).$$

In particular, $\mathbb{R} \subset \mathcal{R}(\mathfrak{so}(p,q))$ coincides with $\mathbb{R}g \wedge g$. Decomposition (2) at the point $x \in M$ has the form

$$R_x = (R_x - H_{0x} \wedge g_x - \mu g_x \wedge g_x) + H_{0x} \wedge g_x + \mu g_x \wedge g_x,$$

where $\mu \in \mathbb{R}$ and H_{0x} belongs to the submodule of $\odot^2 T_xM$ isomorphic to $V_{2\pi_1}$, i.e. H_{0x} is trace-free. Note that the map

$$\mathcal{R}(\mathfrak{so}(p,q)) \to \odot^2\mathbb{R}^{p,q}, \quad R \mapsto \mathrm{Ric}(R), \quad \mathrm{Ric}(R)(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y)$$

is $\mathfrak{so}(p,q)$-equivariant, hence it is zero on the submodule $V_{2\bar{\pi}_2} \subset \mathcal{R}(\mathfrak{so}(p,q))$. Similarly, if $R \in V_{2\pi_1}$, then $g^{ij}\,\mathrm{Ric}(X_i,X_j) = 0$. These conditions imply $H_{0x} = \frac{1}{n-2}\left(\mathrm{Ric}_x - \frac{s_x}{n}g_x\right)$ and $\mu = \frac{s_x}{2n(n-1)}$. Omitting the point x in the above decomposition of R_x, we get Decomposition (2).

If $n=3$, then

$$\mathcal{R}(\mathfrak{so}(p,q)) \simeq V_{4\pi_1} \oplus \mathbb{R}, \tag{5}$$

consequently, $W=0$. If $n=4$, then

$$\mathcal{R}(\mathfrak{so}(p,q)) \simeq (V_{4\pi_1} \oplus V_{4\pi'_1}) \oplus V_{2\pi_1+2\pi'_1} \oplus \mathbb{R}, \tag{6}$$

and W admits a decomposition

$$W = W^+ + W^-. \tag{7}$$

Recall that the Cotton tensor C is defined as follows:

$$C(X,Y,Z) = (n-2)\big(\nabla_Z L(X,Y) - \nabla_Y L(X,Z)\big).$$

It holds

$$(n-3)C(X,Y,Z) = (n-2)g^{ij}\nabla_{X_i} W(Y,Z,X,X_j).$$

If $n=4$, then this and (8) define the decomposition

$$C = C^+ + C^-. \tag{8}$$

3 Decomposition of the covariant derivative of the curvature tensor of a pseudo-Riemannian manifold

For any subalgebra $\mathfrak{h} \subset \mathfrak{so}(p,q)$ define the space

$$\mathcal{R}^\nabla(\mathfrak{h}) = \{S \in \mathrm{Hom}\left(\mathbb{R}^{p,q}, \mathcal{R}(\mathfrak{h})\right) | S_X(Y,Z) + S_Y(Z,X) + S_Z(X,Y) = 0\}.$$

From the second Bianchi identity it follows that if (M,g) is a pseudo-Riemannian manifold with the holonomy algebra $\mathfrak{h}$ at a point $x \in M$, and ∇R is the covariant derivative of the curvature tensor of (M,g), then $\nabla R_x \in \mathcal{R}^\nabla(\mathfrak{h})$.

First we find the space $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$. Let $n = p+q$.

Theorem 1 *The $\mathfrak{so}(p,q)$-module $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ admits the following decomposition into the sum of irreducible $\mathfrak{so}(p,q)$-modules:*

$$\mathcal{R}^\nabla(\mathfrak{so}(p,q)) \simeq V_{6\pi_1} \oplus V_{4\pi_1} \oplus \mathbb{R}^{p,q}, \quad \text{if } n=3, \tag{9}$$

$$\mathcal{R}^\nabla(\mathfrak{so}(p,q)) \simeq (V_{5\pi_1+\pi_1'} \oplus V_{\pi_1+5\pi_1'}) \oplus V_{3\pi_1+3\pi_1'} \oplus (V_{3\pi_1+\pi_1'} \oplus V_{\pi_1+3\pi_1'}) \oplus \mathbb{R}^{p,q}, \quad \text{if } n=4, \tag{10}$$

$$\mathcal{R}^\nabla(\mathfrak{so}(p,q)) \simeq V_{\pi_1+2\bar\pi_2} \oplus V_{3\pi_1} \oplus V_{\pi_1+\bar\pi_2} \oplus \mathbb{R}^{p,q}, \quad \text{if } n \geq 5, \tag{11}$$

where $\bar\pi_2 = 2\pi_2$ if $n=5$ and $\bar\pi_2 = \pi_2$ if $n \geq 6$.

Now we give the explicit form of the above decomposition for the covariant derivative ∇R of the curvature tensor R of a pseudo-Riemannian manifold (M,g) of signature (p,q), $p+q=n$. First we set some notation. Define the following tensors:

$$g(H_X Y, Z) = \frac{1}{3(n+1)}\Big(C(Y,Z,X) + C(Z,Y,X)\Big), \tag{12}$$

$$\begin{aligned} g(T_{0X}Y,Z) =& \frac{1}{3(n-2)}\Big(\nabla_X \mathrm{Ric}(Y,Z) + \nabla_Y \mathrm{Ric}(Z,X) + \nabla_Z \mathrm{Ric}(X,Y) \\ & - \frac{2}{n+2}\big(g(\mathrm{grad}\, s, X)g(Y,Z) + g(\mathrm{grad}\, s, Y)g(Z,X) + g(\mathrm{grad}\, s, Z)g(X,Y)\big)\Big), \end{aligned} \tag{13}$$

$$g(T_{1X}Y,Z) = \frac{1}{2(n-1)(n+2)}\big(g(\mathrm{grad}\, s, X)g(Y,Z) + g(\mathrm{grad}\, s, Y)g(Z,X) + g(\mathrm{grad}\, s, Z)g(X,Y)\big), \tag{14}$$

$$g(P_0(X)Y,Z) = \frac{1}{3(n+1)}C(X,Y,Z), \tag{15}$$

$$(\varphi_1)_X = H_X \wedge g, \tag{16}$$

$$(\varphi_2)_X(Y,Z) = P_0\big((Y\wedge Z)X\big) + X \wedge \big(P_0(Z)Y - P_0(Y)Z\big). \tag{17}$$

In particular, T_0 and T_1 are the symmetrizations of the tensors $\frac{1}{(n-2)}\left(\nabla\operatorname{Ric}-\frac{2}{n+2}sg\right)$ and $\frac{3}{2(n-1)(n+2)}g(\operatorname{grad}s,\cdot)g$, respectively. Note that $T=T_0+T_1$, where

$$g(T_XY,Z)=\frac{1}{3}\left(\nabla_XL(Y,Z)+\nabla_YL(Z,X)+\nabla_ZL(X,Y)\right)$$

is the symmetrization of the tensor ∇L.

Theorem 2 *Let (M,g) be a pseudo-Riemannian manifold of signature (p,q), $p+q=n$. Then the covariant derivative ∇R of the curvature tensor R of (M,g) admits the following decomposition:*

$$\nabla R=S_0'+S_0''+S'+S_1, \tag{18}$$

where

$$S_0'=\nabla W+\frac{3}{n-2}\varphi_1-3\varphi_2, \tag{19}$$

$$(S_0'')_X=T_{0X}\wedge g, \tag{20}$$

$$S'=\varphi_1+3\varphi_2, \tag{21}$$

$$(S_1)_X=T_{1X}\wedge g. \tag{22}$$

If $n=3$, then $S_0'=0$ and $\varphi_1=\varphi_2$. If $n=4$, then S_0' and S' can be further decomposed:

$$S_0'=S_0'^{+}+S_0'^{-},\quad S'=S'^{+}+S'^{-},$$

where $S_0'^{\pm}$ and $S'^{\pm}$ are given by the same formulas as S_0' and S', respectively, with W and C replaced by $W^{\pm}$ and $C^{\pm}$.

The fact that S_0' satisfies the second Bianchi identity implies the known second Bianchi identity for the tensor W [7], we may rewrite it in the form

$$\begin{aligned} g\big((\nabla_XW(Y,Z)&+\nabla_YW(Z,X)+\nabla_ZW(X,Y))V,U\big)\\ &=-\frac{1}{n-2}\Big(C(U,X,Y)g(Z,V)+C(U,Y,Z)g(X,V)+C(U,Z,X)g(Y,V)\\ &\qquad -C(V,X,Y)g(Z,U)-C(V,Y,Z)g(X,U)-C(V,Z,X)g(Y,U)\Big). \end{aligned}\tag{23}$$

Note that

$$(S_0''+S_1)_X=T_X\wedge g.$$

Consider some traces for the obtained tensors. The tensor S_0' is totally trace-free. It holds

$$(S_0'')_{X_i}(Z,X_j)g^{ij}=0, \tag{24}$$

$$(S')_{X_i}(X,Y,Z,X_j)g^{ij}=C(Z,X,Y), \tag{25}$$

$$(S_1)_{X_i}(X,Y,Z,X_j)g^{ij}=\frac{1}{2(n-1)}\big(g(\operatorname{grad}s,X)g(Y,Z)-g(\operatorname{grad}s,Y)g(X,Z)\big), \tag{26}$$

$$(S_0'')_X(X_i,Y,X_j,Z)g^{ij}=(n-2)g(T_{0X}Y,Z), \tag{27}$$

$$(S')_X(X_i,Y,X_j,Z)g^{ij}=\frac{1}{3}\big(C(Y,Z,X)+C(Z,Y,X)\big), \tag{28}$$

$$(S_1)_X(X_i,Y,X_j,Z)g^{ij}=\frac{1}{2(n-1)}g(\operatorname{grad}s,X)g(Y,Z)+(n-2)g(T_{1X}Y,Z). \tag{29}$$

In [9, 3] it is shown that the space $\mathcal{R}ic_x^{\nabla}$ of possible values of the tensor $\nabla \operatorname{Ric}_x$ admits a decomposition

$$\mathcal{R}ic_x^{\nabla} = Q_x \oplus S_x \oplus A_x.$$

into the sum of irreducible $\mathfrak{so}(T_xM)$-modules. This corresponds to the decomposition

$$\nabla \operatorname{Ric} = \xi_Q + \xi_S + \xi_A.$$

It can be checked that

$$\xi_Q = \frac{1}{2(n-1)} g(\operatorname{grad} s, \cdot) g + (n-2)T_1, \quad \xi_S = (n-2)T_0, \quad (\xi_A)_X(Y,Z) = \frac{1}{3}\big(C(Y,Z,X) + C(Z,Y,X)\big).$$

This shows that the decomposition of $\nabla \operatorname{Ric}$ from [9, 3] can be obtained using the above decomposition of ∇R taking $\operatorname{tr}_{(2,4)}$.

4 Decomposition of the curvature tensor of a pseudo-Kählerian manifold

Let (M, g, J) be a pseudo-Kählerian manifold of signature $(2p, 2q)$. Let $n = p + q \geq 2$. In order to simplify the usual expression for the decomposition of the curvature tensor of (M, g, J) we set some notation, similar ideas can be found in [2]. The tangent space to (M, g, J) is identified with the pseudo-Euclidean space $\mathbb{R}^{2p,2q}$ endowed with a pseudo-Euclidean metric g and a g-orthogonal complex structure J. Using J, we may identify $\mathbb{R}^{2p,2q}$ with $\mathbb{C}^n$ in such a way that J corresponds to the multiplication by the complex unit i. Consider the corresponding pseudo-Hermitian metric

$$h(X,Y) = g(X,Y) + g(X,JY)J, \quad X, Y \in \mathbb{R}^{2p,2q}.$$

The expression

$$(X \wedge_J Y)Z = h(Z,X)Y - h(Z,Y)X \tag{30}$$

defines an element $X \wedge_J Y \in \mathfrak{u}(p,q)$. Note that

$$X \wedge_J Y = X \wedge Y + JX \wedge JY.$$

This construction allows to identify $\mathfrak{u}(p,q)$ with the space

$$\wedge_J^2 \mathbb{R}^{2p,2q} = \operatorname{span}\{X \wedge_J Y | X, Y \in \mathbb{R}^{2p,2q}\} \subset \wedge^2 \mathbb{R}^{2p,2q}.$$

Under this identification, the complex structure $J \in \mathfrak{u}(p,q)$ corresponds to the element

$$\frac{1}{2}\sum_{i=1}^{n} \epsilon_i e_i \wedge_J Je_i,$$

where $e_1, ..., e_n, Je_1, ..., Je_n$ is an orthonormal basis (i.e. it is an orthogonal basis such that $g(e_i, e_i) = \epsilon_i$, where $\epsilon_i = -1$ for $i = 1, ..., p$ and $\epsilon_i = 1$ for $i = p+1, ..., n$). The complex space $\wedge^2 \mathbb{C}^n$ may be considered as the following subspace of $\wedge^2 \mathbb{R}^{2p,2q}$:

$$\wedge^2 \mathbb{C}^n = \{X \wedge Y - JX \wedge JY | X, Y \in \mathbb{R}^{2p,2q}\}.$$

We obtain the orthogonal decomposition

$$\wedge^2\mathbb{R}^{2p,2q} = \wedge^2_J\mathbb{R}^{2p,2q} \oplus \wedge^2\mathbb{C}^n = \mathfrak{u}(p,q) \oplus \wedge^2\mathbb{C}^n.$$

Let $R \in \mathcal{R}(\mathfrak{u}(p,q))$. Then

$$R : \wedge^2\mathbb{R}^{2p,2q} \to \mathfrak{u}(p,q) \subset \wedge^2\mathbb{R}^{2p,2q}$$

is a symmetric linear map. Consequently, R is zero on the orthogonal complement to $\mathfrak{u}(p,q)$ in $\wedge^2\mathbb{R}^{2p,2q}$, i.e. R is zero on $\wedge^2\mathbb{C}^n$. This implies the known equalities

$$R(JX,JY) = R(X,Y), \quad R(JX,Y) + R(X,JY) = 0. \tag{31}$$

Note that

$$R(X \wedge_J Y) = R(X \wedge Y) + R(JX \wedge JY) = 2R(X \wedge Y). \tag{32}$$

Note also that

$$R(X \wedge Y) = R(X \otimes Y - Y \otimes X) = 2R(X,Y).$$

Thus,

$$R(X,Y) = R(X \wedge_J Y). \tag{33}$$

From [1] it follows that the space $\mathcal{R}(\mathfrak{u}(p,q))$ admits the following decomposition:

$$\mathcal{R}(\mathfrak{u}(p,q)) = \mathcal{R}_0(\mathfrak{u}(p,q)) \oplus \mathcal{R}'(\mathfrak{u}(p,q)) \oplus \mathcal{R}_1(\mathfrak{u}(p,q)),$$

where $\mathcal{R}_0(\mathfrak{u}(p,q)) \otimes \mathbb{C} \simeq V_{2\pi_1+2\pi_{n-1}}$ and

$$\mathcal{R}'(\mathfrak{u}(p,q)) \oplus \mathcal{R}_1(\mathfrak{u}(p,q)) \simeq \mathfrak{su}(p,q) \oplus \mathbb{R}J = \mathfrak{u}(p,q).$$

Here by V_Λ we denote the irreducible representations of the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ with the highest weight Λ. To describe the above isomorphism, note that if $A \in \mathfrak{u}(p,q)$, then $H = JA \in \odot^2\mathbb{R}^{2p,2q}$ is a symmetric endomorphism commuting with J, or, equivalently, H defines a symmetric bilinear form such that $H(JX,Y) + H(X,JY) = 0$. Any H with such property may be obtained in this way. Next, such H defines $R_H \in \mathcal{R}(\mathfrak{u}(p,q))$ given by

$$R_H(X,Y) = HX \wedge_J Y + X \wedge_J HY + 2g(HJX,Y) + 2g(JX,Y)JH.$$

We set the notation

$$H \wedge_J g = R_H.$$

Now, $\mathcal{R}'(\mathfrak{u}(p,q))$ is spanned by elements R_H with trace-free H, and $J \in \mathfrak{u}(p,q)$ defines the element $-\operatorname{Id}\wedge_J g \in \mathcal{R}_1(\mathfrak{u}(p,q))$. Note that

$$\left(\frac{\operatorname{Id}}{2} \wedge_J g\right)(X,Y) = X \wedge_J Y + 2g(JX,Y)J.$$

The above decomposition of $\mathcal{R}(\mathfrak{u}(p,q))$ shows that any $R \in \mathcal{R}(\mathfrak{u}(p,q))$ may be written in the form

$$R = B + H \wedge_J g + \lambda\frac{\operatorname{Id}}{2} \wedge_J g,$$

where B is a totally trace-free tensor, H is trace-free and $\lambda \in \mathbb{R}$. It can be computed that $H = -\frac{1}{2(n+2)}\operatorname{Ric}^0$ and $\lambda = -\frac{s}{4n(n+1)}$, here $\operatorname{Ric}^0 = \operatorname{Ric} - \frac{s}{2n}g$ is the trace-free part of the Ricci tensor Ric defined by R, and s is the trace of Ric, i.e. the scalar curvature defined by R.

Let (M, g, J) be a pseudo-Kählerian manifold. Then its curvature tensor admits the decomposition:

$$R = B - \frac{1}{2(n+2)}\left(\mathrm{Ric} - \frac{s}{2n}g\right) \wedge_J g - \frac{s}{4n(n+1)}\frac{\mathrm{id}}{2} \wedge_J g. \tag{34}$$

The tensor B is called the Bochner curvature tensor, and it is an analog of the Weyl conformal tensor W. The above equality can be rewritten in the form

$$R = B + K \wedge_J g, \tag{35}$$

where

$$K = -\frac{1}{2(n+2)}\left(\mathrm{Ric} - \frac{s}{4(n+1)}g\right)$$

is the analog of the Schouten tensor L.

Define the tensor D in the following way:

$$D(Z, X, Y) = g(g^{ab}\nabla_{X_a}R(Z, X_b)Y, X) - \frac{1}{4(n+1)}g\left(\left(\frac{\mathrm{id}}{2} \wedge_J g\right)(\mathrm{grad}\, s, Z)Y, X\right). \tag{36}$$

The tensor D is the analog of the Cotton tensor C. The following holds [17]:

$$D(Z, X, Y) = \frac{n+2}{n}g(g^{ab}\nabla_{X_a}B(Z, X_b)Y, X). \tag{37}$$

Let

$$g(\tilde{D}(X)Y, Z) = D(X, Y, Z), \tag{38}$$

then

$$\tilde{D}(Z) = -g^{ab}\nabla_{X_a}R(Z, X_b) + \frac{1}{4(n+1)}\left(\frac{\mathrm{id}}{2} \wedge_J g\right)(\mathrm{grad}\, s, Z). \tag{39}$$

5 Decomposition of the covariant derivative of the curvature tensor of a pseudo-Kählerian manifold

Let W be an irreducible $\mathfrak{u}(p, q)$-module such that J acts as a complex structure on W. Then the $\mathfrak{gl}(n, \mathbb{C})$-module $W \otimes \mathbb{C}$ decomposes into the direct sum $V \oplus \bar{V}$ of irreducible $\mathfrak{gl}(n, \mathbb{C})$-modules, where V and $\bar{V}$ are the eigenspaces of the extension of J to $W \otimes \mathbb{C}$ corresponding to the eigenvalues i and $-$i, respectively. The representation of $\mathfrak{gl}(n, \mathbb{C})$ on $V \oplus \bar{V}$ is given in the matrix form

$$\left\{ \left(\begin{array}{cc} \rho(A) & 0 \\ 0 & -\overline{\rho(A)}^t \end{array} \right) \middle| A \in \mathfrak{gl}(n, \mathbb{C}) \right\}, \tag{40}$$

where $\rho : \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(V)$ is the representation of $\mathfrak{gl}(n, \mathbb{C})$ on V. Note that W is isomorphic to V as the complex vector space. The vector space V is the highest weight $\mathfrak{gl}(n, \mathbb{C})$-module with a highest weight Λ, and we denote V by V_Λ. Thus, $W \simeq V_\Lambda$. Next, if $n \geq 3$ and V_Λ is an irreducible representation of $\mathfrak{gl}(n, \mathbb{C})$ with the highest weight Λ and the labels of Λ are situated not symmetrically on the Dynkin diagram of $\mathfrak{sl}(n, \mathbb{C})$, then the restriction of this representation to $\mathfrak{u}(p, q)$ is irreducible [19].

The space $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$ admits the natural complex structure,

$$(J\cdot S)_X = S_{-JX}$$

given by the representation of $\mathfrak{u}(p,q)$ on $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$. Hence, each irreducible component of $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$ is of the form V_Λ.

Theorem 3 *If $n\geq 3$, then the $\mathfrak{u}(p,q)$-module $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$ admits the following decomposition into the sum of irreducible $\mathfrak{u}(p,q)$-modules:*

$$\mathcal{R}^\nabla(\mathfrak{u}(p,q)) = \mathcal{Q}_0\oplus\mathcal{Q}'\oplus\mathcal{Q}_1 \simeq V_{2\pi_1+\pi_{n-1}}\oplus V_{\pi_2+\pi_{n-1}}\oplus\mathbb{R}^{2p,2q}.$$

If $n=2$, then

$$\mathcal{R}^\nabla(\mathfrak{u}(p,q)) = \mathcal{Q}_0\oplus\mathcal{Q}'\oplus\mathcal{Q}_1 \simeq \mathbb{C}^6\oplus\mathbb{C}^4\oplus\mathbb{R}^{2p,2q}.$$

Note that

$$\mathcal{Q}'\simeq\mathcal{P}_0(\mathfrak{u}(p,q))\simeq(\odot^2(\mathbb{C}^n)^*\otimes\mathbb{C}^n)_0,$$

where $(\odot^2(\mathbb{C}^n)^*\otimes\mathbb{C}^n)_0$ is the subspace of $\odot^2(\mathbb{C}^n)^*\otimes\mathbb{C}^n$ consisting of tensors such that the contraction of the upper index with any down index gives zero. The explicit isomorphism will be constructed in the proof of Theorem 3.

Recall that we consider a pseudo-Kählerian manifold (M,g,J) of signature $(2p,2q)$, $p+q=n\geq 2$. Now we give the explicit form of the above decomposition for the covariant derivative ∇R of the curvature tensor R of (M,g,J). Let us first set some notation. Define the following tensors:

$$T_X = \frac{1}{8(n+2)(n+1)}\Big(J\circ(X\wedge_J J\,\mathrm{grad}\,s) - g(\mathrm{grad}\,s,X)\,\mathrm{Id}\Big), \tag{41}$$

$$(\psi_1)_X = -\frac{1}{2(n+3)}J\tilde{D}(JX)\wedge_J g, \tag{42}$$

$$(\psi_2)_X(Y,Z) = -\frac{1}{2(n+3)}\left(\tilde{D}\big((Y\wedge_J Z)X\big) + X\wedge_J\left(\tilde{D}(Z)Y-\tilde{D}(Y)Z\right)\right). \tag{43}$$

Theorem 4 *Let (M,g,J) be a pseudo-Kählerian manifold of signature $(2p,2q)$, $p+q=n\geq 2$. Then the covariant derivative ∇R of the curvature tensor R of (M,g,J) admits the following decomposition:*

$$\nabla R = Q_0 + Q' + Q_1, \tag{44}$$

where

$$Q_0 = \nabla B + \frac{1}{n+2}\psi_1 + \psi_2, \tag{45}$$

$$Q' = \psi_1 - \psi_2, \tag{46}$$

$$(Q_1)_X = T_X\wedge_J g. \tag{47}$$

The fact that Q_0 satisfies the second Bianchi identity can be expressed in the form

$$\begin{aligned} g\big((\nabla_X B(Y,Z)+\nabla_Y B(Z,X)+\nabla_Z B(X,Y))V,U\big) \\ = \frac{1}{2(n+2)}\Big(&D(U,X,Y)h(V,Z)+D(U,Y,Z)h(V,X)+D(U,Z,X)h(V,Y) \\ &+D(h(Z,U),Y,X)+D(h(X,U),Z,Y)+D(h(Y,U),X,Z) \\ &-2D(g(JX,Y)JZ+g(JY,Z)JX+g(JZ,X)JY,V,U)\Big). \end{aligned} \tag{48}$$

This gives the second Bianchi identity for the Bochner tensor B, similar to the second Bianchi identity for the Weyl tensor (23). This identity written using the index notation is proved directly in [15].

Consider some traces for the obtained tensors. The tensor Q_0 is totally trace-free. It holds

$$(Q')_{X_a}(X,Y,Z,X_b)g^{ab} = D(Z,X,Y), \tag{49}$$

$$(Q_1)_{X_a}(X,Y,Z,X_b)g^{ab} = \frac{1}{4(n+1)}g\left(\left(\frac{\mathrm{id}}{2}\wedge_J g\right)(\operatorname{grad} s,Z)Y,X\right), \tag{50}$$

$$(Q')_X(X_a,Y,X_b,Z)g^{ab} = -D(JX,JZ,Y), \tag{51}$$

$$(Q_1)_X(X_a,Y,X_b,Z)g^{ab} = \frac{1}{4(n+1)}g\left(\left(\frac{\mathrm{id}}{2}\wedge_J g\right)(X,J\operatorname{grad} s)JY,Z\right). \tag{52}$$

This shows that

$$\nabla_X \operatorname{Ric}(Y,Z) = -D(JX,JZ,Y) + \frac{1}{4(n+1)}g\left(\left(\frac{\mathrm{id}}{2}\wedge_J g\right)(X,J\operatorname{grad} s)JY,Z\right),$$

and the space of values of the covariant derivatives of the Ricci tensor is decomposed into the direct sum of two irreducible $\mathfrak{u}(p,q)$-modules isomorphic, respectively, to $(\odot^2(\mathbb{C}^n)^*\otimes\mathbb{C}^n)_0$ and $\mathbb{R}^{2p,2q}$.

We also get

$$\nabla_X \operatorname{Ric}(JY,Z) = D(JX,Y,Z) - \frac{1}{4(n+1)}g\left(\left(\frac{\mathrm{id}}{2}\wedge_J g\right)(X,J\operatorname{grad} s)Y,Z\right),$$

this corresponds to the fact that $\nabla\operatorname{Ric}(J\cdot,\cdot)$ satisfies

$$\nabla_X \operatorname{Ric}(JY,Z) + \nabla_Y \operatorname{Ric}(JZ,X) + \nabla_Z \operatorname{Ric}(JX,Y) = 0,$$

i.e. $\nabla\operatorname{Ric}(J\cdot,\cdot)\in\mathcal{P}(\mathfrak{u}(TM))$ and gives the decomposition of $\nabla\operatorname{Ric}(J\cdot,\cdot)$ corresponding to the decomposition $\mathcal{P}(\mathfrak{u}(p,q)) = \mathcal{P}_0(\mathfrak{u}(p,q))\oplus\mathcal{P}_1(\mathfrak{u}(p,q))$ (see below).

6 Proof of Theorem 1

Consider a subalgebra $\mathfrak{h}\subset\mathfrak{so}(p,q)$ and its complexification $\mathfrak{h}\otimes\mathbb{C}\subset\mathfrak{so}(\mathbb{R}^{p,q}\otimes\mathbb{C})$. We will use the facts that

$$\mathcal{R}(\mathfrak{h}\otimes\mathbb{C}) = \mathcal{R}(\mathfrak{h})\otimes\mathbb{C}, \qquad \mathcal{R}^\nabla(\mathfrak{h}\otimes\mathbb{C}) = \mathcal{R}^\nabla(\mathfrak{h})\otimes\mathbb{C}. \tag{53}$$

Let V_Λ denote the irreducible $\mathfrak{so}(n,\mathbb{C})$-module of $\mathfrak{so}(n,\mathbb{C})$ with the highest weight Λ. If the restriction of this representation of $\mathfrak{so}(n,\mathbb{C})$ to $\mathfrak{so}(p,q)\subset\mathfrak{so}(n,\mathbb{C})$ is reducible (this happens if and only if $n\geq 4m+2$ or $n=4m+2$ and the labels of Λ on the Dynkin diagram of $\mathfrak{so}(4m+2,\mathbb{C})$ are situated symmetrically), then as $\mathfrak{so}(p,q)$-module V_Λ decomposes in the direct sum $V_\Lambda^{\mathfrak{so}(p,q)}\oplus iV_\Lambda^{\mathfrak{so}(p,q)}$ for some irreducible $\mathfrak{so}(p,q)$-module $V_\Lambda^{\mathfrak{so}(p,q)}$. When it does not lead to confuse, we denote $V_\Lambda^{\mathfrak{so}(p,q)}$ simply by V_Λ (this denotation is used in Sections 2 and 3).

First suppose that $n\geq 5$. Using (4), we get

$$\begin{aligned}\mathcal{R}^\nabla(\mathfrak{so}(p,q)) \subset \mathbb{R}^{p,q}\otimes\mathcal{R}(\mathfrak{so}(p,q)) &= \mathbb{R}^{p,q}\otimes(V_{2\bar\pi_2}\oplus V_{2\pi_1}\oplus\mathbb{R})\\ &= \left(V_{\pi_1+2\bar\pi_2}\oplus V_\lambda\oplus V_{\pi_1+\bar\pi_2}\right)\oplus\left(V_{3\pi_1}\oplus V_{\pi_1+\bar\pi_2}\oplus\mathbb{R}^{p,q}\right)\oplus\mathbb{R}^{p,q},\end{aligned} \tag{54}$$

where $\lambda = 4\pi_2$ if $n = 5$, and $\lambda = \pi_2 + \pi_3$ if $n \geq 6$. Consider the $\mathfrak{so}(n, \mathbb{C})$-submodule $V_{\pi_1+2\bar{\pi}_2} \subset \mathbb{C}^n \otimes V_{2\bar{\pi}_2}$. This submodule is the highest one. Let $v_{-m}, ..., v_{-1}, v_1, ..., v_m$ and $v_{-m}, ..., v_{-1}, v_0, v_1, ..., v_m$ be the standard bases of $\mathbb{C}^{2m}$ and $\mathbb{C}^{2m+1}$, respectively. A highest weight vector of the $\mathfrak{so}(n, \mathbb{C})$-module $V_{2\bar{\pi}_2}$ has the form $(v_1 \wedge v_2) \odot (v_1 \wedge v_2)$ [1]. Hence,

$$\xi = v_1 \otimes ((v_1 \wedge v_2) \odot (v_1 \wedge v_2))$$

is a highest weight vector of the module $V_{\pi_1+2\bar{\pi}_2}$. Clearly, $\xi \in \mathcal{R}^\nabla(\mathfrak{so}(n, \mathbb{C}))$, consequently, $V_{\pi_1+2\bar{\pi}_2} \subset \mathcal{R}^\nabla(\mathfrak{so}(n, \mathbb{C}))$. Suppose that $n \geq 6$. Consider the submodule

$$V_{\pi_2+\pi_3} \subset \mathbb{C}^n \otimes V_{2\pi_2} = V_{\pi_1+2\pi_2} \oplus V_{\pi_2+\pi_3} \oplus V_{\pi_1+\pi_2}.$$

Consider the element

$$\xi = v_3 \otimes ((v_1 \wedge v_2) \odot (v_1 \wedge v_2)) \in \mathbb{C}^n \otimes V_{2\pi_2},$$

which has weight $\pi_2 + \pi_3$. It is easy to check that $\xi \notin \mathcal{R}^\nabla(\mathfrak{so}(n, \mathbb{C}))$. Note that the module $V_{\pi_1+\pi_2}$ does not contain the weight space of weight $\pi_2 + \pi_3$. Hence, there exist $\xi_1 \in V_{\pi_1+2\pi_2}$ and $\xi_2 \in V_{\pi_2+\pi_3}$ such that $\xi = \xi_1 + \xi_2$. We obtain that $\xi_2 \neq 0$ and $\xi_2 \notin \mathcal{R}^\nabla(\mathfrak{so}(n, \mathbb{C}))$ this shows that $V_{\pi_2+\pi_3} \not\subset \mathcal{R}^\nabla(\mathfrak{so}(n, \mathbb{C}))$. Similarly, if $n = 5$, then $V_{4\pi_2} \not\subset \mathcal{R}^\nabla(\mathfrak{so}(n, \mathbb{C}))$.

For any linear map $S : \mathbb{R}^{p,q} \to \mathcal{R}(\mathfrak{so}(p, q))$ define the map

$$P : \mathbb{R}^{p,q} \to \mathfrak{so}(p, q), \quad P(X) = \operatorname{tr}_{(1,5)} S.(\cdot, \cdot, X, \cdot) = g^{ij} S_{X_i}(X, X_j). \tag{55}$$

It is easy to check that for any $S \in \mathcal{R}^\nabla(\mathfrak{so}(p, q))$, the element $\operatorname{tr}_{(1,5)} S$ belongs to $\mathcal{P}(\mathfrak{so}(p, q))$. For any subalgebra $\mathfrak{h} \subset \mathfrak{so}(p, q)$ and $P \in \mathcal{P}(\mathfrak{h})$ define the vector

$$\widetilde{\operatorname{Ric}}(P) = g^{ij} P(X_i)X_j \in \mathbb{R}^{p,q}. \tag{56}$$

In [8] it is shown that the space $\mathcal{P}(\mathfrak{h})$ admits the decomposition

$$\mathcal{P}(\mathfrak{h}) = \mathcal{P}_0(\mathfrak{h}) \oplus \mathcal{P}_1(\mathfrak{h}),$$

where $\mathcal{P}_0(\mathfrak{h}) = \ker \widetilde{\operatorname{Ric}}$ and $\mathcal{P}_1(\mathfrak{h})$ is its orthogonal complement.

It holds

$$\mathcal{P}_0(\mathfrak{so}(p, q)) \simeq V_{\pi_1+\bar{\pi}_2}, \quad \mathcal{P}_1(\mathfrak{so}(p, q)) \simeq \mathbb{R}^{p,q}.$$

In correspondence with this decomposition, any tensor $P \in \mathcal{P}(\mathfrak{so}(p, q))$ can be decomposed in the sum

$$P = P_0 + P_1, \quad P_0(X) = P(X) + \frac{1}{n-1}\widetilde{\operatorname{Ric}}(P) \wedge X, \quad P_1(X) = -\frac{1}{n-1}\widetilde{\operatorname{Ric}}(P) \wedge X. \tag{57}$$

For any tensor P define the tensors $\Phi_1(P)$ and $\Phi_2(P)$ as follows:

$$\Phi_1(P)_X(Y, Z) = H_X Y \wedge Z + Y \wedge H_X Z, \tag{58}$$

where $B : \mathbb{R}^{p,q} \to \odot^2\mathbb{R}^{p,q}$ is given by

$$g(H_X Y, Z) = g(P(Y)Z + P(Z)Y, X),$$

and

$$\Phi_2(P)_X(Y,Z) = P\big((Y\wedge Z)X\big) + X\wedge (P(Z)Y - P(Y)Z). \tag{59}$$

It can be checked that $\Phi_1(P)$ and $\Phi_2(P)$ map $\mathbb{R}^{p,q}$ to $\mathcal{R}(\mathfrak{so}(p,q)$, and the maps Φ_1 and Φ_2 are $\mathfrak{so}(p,q)$-equivariant. Let $\alpha,\beta\in\mathbb{R}$ and $P\neq 0$. It can be shown that $(\alpha\Phi_1+\beta\Phi_2)(P)$ belongs to $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ if and only if $\beta=3\alpha$. Next,

$$\mathrm{tr}_{(1,5)}(\Phi_1(P_0)) = -3P_0,\quad \mathrm{tr}_{(1,5)}(\Phi_1(P_1)) = (n-4)P_1,\quad \mathrm{tr}_{(1,5)}(\Phi_2(P)) = -nP, \tag{60}$$

where $P\in\mathcal{P}(\mathfrak{so}(p,q))$, $P_0\in\mathcal{P}_0(\mathfrak{so}(p,q))$ and $P_1\in\mathcal{P}_1(\mathfrak{so}(p,q))$. This shows that $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ contains exactly one submodule isomorphic to $V_{\pi_1+\bar\pi_2}$ and exactly one submodule isomorphic to $\mathbb{R}^{p,q}$.

Recall that $\mathcal{R}(\mathfrak{so}(p,q))$ contains the submodule $\odot^2\mathbb{R}^{p,q}$, hence $\mathbb{R}^{p,q}\otimes\mathcal{R}(\mathfrak{so}(p,q))$ contains the submodule $\odot^3\mathbb{R}^{p,q}\simeq V_{3\pi_1}\oplus\mathbb{R}^{p,q}$. The inclusion $\odot^3\mathbb{R}^{p,q}\hookrightarrow\mathbb{R}^{p,q}\otimes\mathcal{R}(\mathfrak{so}(p,q))$ is given by

$$T\mapsto S,\quad S_X = T_X\wedge g.$$

Clearly, $S\in\mathcal{R}^\nabla(\mathfrak{so}(p,q))$. This shows that $V_{3\pi_1}\subset\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ and it gives the explicit inclusion $\odot^3\mathbb{R}^{p,q}\hookrightarrow\mathcal{R}^\nabla(\mathfrak{so}(p,q))$. For $n=3$ and 4 the proof is similar. We have proved Theorem 1. $\square$

7 Proof of Theorem 2

Let (M,g) be a pseudo-Riemannian manifold. Suppose that $n\geq 6$. From Theorem 1 it follows that

$$\nabla R = S_0' + S_0'' + S' + S_1$$

for some tensors S_0', S_0'', S', S_1 such that the values of these tensors at any $x\in M$ belong to the submodules of $\mathcal{R}^\nabla(\mathfrak{so}(T_xM))$ isomorphic respectively to $V_{\pi_1+2\pi_2}$, $V_{3\pi_1}$, $V_{\pi_1+\pi_2}$ and $\mathbb{R}^{p,q}$.

From (3) it follows that

$$\nabla_X R = \nabla_X W + \nabla_X L\wedge g.$$

Let $x\in M$. Then for each $X\in T_xM$, $\nabla_X W_x$ belongs to $V_{2\pi_2}\subset\mathcal{R}(\mathfrak{so}(p,q))$, while $\nabla_X L_x\wedge g_x$ belongs to $\odot^2\mathbb{R}^{p,q} = V_{2\pi_1}\oplus\mathbb{R}\subset\mathcal{R}(\mathfrak{so}(p,q))$. We get

$$\nabla W_x \in \mathbb{R}^{p,q}\otimes V_{2\pi_2} = V_{\pi_1+2\pi_2}\oplus V_{\pi_1+\pi_2}\oplus V_{\pi_2+\pi_3}, \tag{61}$$

$$\nabla L_x \in \mathbb{R}^{p,q}\otimes (V_{2\pi_1}\oplus\mathbb{R}) = (V_{3\pi_1}\oplus V_{\pi_1+\pi_2}\oplus\mathbb{R}^{p,q})\oplus\mathbb{R}^{p,q}. \tag{62}$$

Since $\mathcal{R}^\nabla(\mathfrak{so}(p,q))$ does not contain the submodule $V_{\pi_2+\pi_3}$, we obtain

$$\nabla W_x \in V_{\pi_1+2\pi_2}\oplus V_{\pi_1+\pi_2}. \tag{63}$$

From the above we get that there are unique elements $P_{0x}\in\mathcal{P}_0(\mathfrak{so}(p,q))$ and $P_{1x}\in\mathcal{P}_1(\mathfrak{so}(p,q))$ such that

$$\mathrm{tr}_{(1,5)}(\nabla R_x - (\Phi_1+3\Phi_2)(P_{0x}+P_{1x})) = 0.$$

Let us find these elements. Using the second Bianchi identity, it can be shown that

$$g^{ij}\nabla_{X_i}R(X,Y,Z,X_j) = C(Z,X,Y) + \frac{1}{2(n-1)}g(\mathrm{grad}\, s, (X\wedge Y)Z).$$

From this and (60) we get

$$g(P_0(X)Y,Z) = \frac{1}{3(n+1)}C(X,Y,Z), \quad P_1(X) = -\frac{1}{4(n+2)(n-1)}\,\mathrm{grad}\, s \wedge X.$$

Let $P = P_0 + P_1$. We obtain

$$\nabla R_x - (\Phi_1 + 3\Phi_2)(P_x) \in V_{\pi_1+2\pi_2} \oplus V_{3\pi_1}.$$

This means that

$$S_1 = (\Phi_1 + 3\Phi_2)(P_1), \quad S' = (\Phi_1 + 3\Phi_2)(P_0), \quad S_0' + S_0'' = \nabla R - (\Phi_1 + 3\Phi_2)(P).$$

Note that $\Phi_1(P_0) = \varphi_1$ and $\Phi_2(P_0) = \varphi_2$. Recall that $\odot^3\mathbb{R}^{p,q} \simeq V_{3\pi_1} \oplus \mathbb{R}^{p,q}$. This, Theorem 1 and (62) show that

$$(S_0'' + S_1)_X = T_X \wedge g,$$

where

$$g(T_X Y, Z) = \frac{1}{3}\left(\nabla_X L(Y,Z) + \nabla_Y L(Z,X) + \nabla_Z L(X,Y)\right)$$

is the symmetrization of ∇L. Using this and the expression for S_1, we find S_0''. Finally we find S_0' from the equality

$$S_0' = \nabla W + \nabla L \wedge g - S_0'' - S_1 - S'.$$

Theorem 2 is proved. $\square$

8 Proof of Theorem 3

First consider the complexification of the space $\mathcal{R}(\mathfrak{u}(p,q))$. As it is noted above, the representation of $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u}(p,q) \otimes \mathbb{C}$ on the space $\mathbb{R}^{p,q} \otimes \mathbb{C}$ decomposes into the sum $V \oplus \bar{V} = V_{\pi_1} \oplus V_{\pi_{n-1}}$ and the representation is of the form (40). From this, (53) and the Bianchi identity it follows that any $R \in \mathcal{R}(\mathfrak{u}(p,q)) \otimes \mathbb{C}$ satisfies $R(V,V) = R(\bar{V},\bar{V}) = 0$. Consequently,

$$\mathcal{R}(\mathfrak{u}(p,q)) \otimes \mathbb{C} \subset (V_{\pi_1} \oplus V_{\pi_{n-1}}) \otimes \mathfrak{gl}(n,\mathbb{C}).$$

More precisely,

$$\mathcal{R}(\mathfrak{u}(p,q)) \otimes \mathbb{C} = V_{2\pi_1+2\pi_{n-1}} \oplus \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}.$$

Hence,

$$\begin{aligned} \mathcal{R}^\nabla(\mathfrak{u}(p,q)) \otimes \mathbb{C} &\subset (V_{\pi_1} \oplus V_{\pi_{n-1}}) \otimes (\mathcal{R}(\mathfrak{u}(p,q)) \otimes \mathbb{C}) \\ &= (V_{3\pi_1+2\pi_{n-1}} \oplus V_{\pi_1+\pi_2+2\pi_{n-1}} \oplus 2V_{2\pi_1+\pi_{n-1}} \oplus 2V_{\pi_1} \oplus V_{\pi_2+\pi_{n-1}}) \\ &\quad \oplus (V_{2\pi_1+3\pi_{n-1}} \oplus V_{2\pi_1+\pi_{n-2}+\pi_{n-1}} \oplus 2V_{\pi_1+2\pi_{n-1}} \oplus 2V_{\pi_{n-1}} \oplus V_{\pi_1+\pi_{n-2}}). \end{aligned} \tag{64}$$

Since $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$ admits a complex structure, each irreducible submodule of $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$ defines two $\mathfrak{gl}(n,\mathbb{C})$-irreducible submodules of $\mathcal{R}^\nabla(\mathfrak{u}(p,q)) \otimes \mathbb{C}$.

Consider the $\mathfrak{gl}(n,\mathbb{C})$-submodule $V_{3\pi_1+2\pi_{n-1}} \subset \mathbb{C}^n \otimes V_{2\pi_1+2\pi_{n-1}}$. This submodule is the highest one. Let $v_1, ..., v_n$ and $v_{-1}, ..., v_{-n}$ be the basis of $V = V_{\pi_1} = \mathbb{C}^n$ and the dual basis of $\bar{V} =$

$V_{\pi_{n-1}} = (\mathbb{C}^n)^*$, respectively. A highest weight vector of the $\mathfrak{gl}(n,\mathbb{C})$-module $V_{2\pi_1+2\pi_{n-1}}$ has the form $(v_1 \otimes v_{-n}) \otimes (v_1 \otimes v_{-n})$. Hence,

$$\xi = v_1 \otimes (v_1 \otimes v_{-n}) \otimes (v_1 \otimes v_{-n})$$

is a highest weight vector of the module $V_{3\pi_1+2\pi_{n-1}}$. Clearly, $\xi \in \mathcal{R}^\nabla(\mathfrak{u}(p,q)) \otimes \mathbb{C}$, consequently, $V_{3\pi_1+2\pi_{n-1}} \subset \mathcal{R}^\nabla(\mathfrak{u}(p,q)) \otimes \mathbb{C}$. Similarly, $V_{2\pi_1+3\pi_{n-1}} \subset \mathcal{R}^\nabla(\mathfrak{u}(p,q)) \otimes \mathbb{C}$. These two modules define an irreducible submodule of $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$ that we denote by $\mathcal{Q}_0$. Consider the submodule

$$V_{\pi_1+\pi_2+2\pi_{n-1}} \subset \mathbb{C}^n \otimes V_{2\pi_1+2\pi_{n-1}} = V_{3\pi_1+2\pi_{n-1}} \oplus V_{\pi_1+\pi_2+2\pi_{n-1}} \oplus V_{2\pi_1+\pi_{n-1}}.$$

Consider the element

$$\xi = v_2 \otimes (v_1 \otimes v_{-n}) \otimes (v_1 \otimes v_{-n}) \in \mathbb{C}^n \otimes V_{2\pi_1+2\pi_{n-1}},$$

which has weight $\pi_1 + \pi_2 + 2\pi_{n-1}$. It is easy to check that $\xi \notin \mathcal{R}^\nabla(\mathfrak{u}(p,q)) \otimes \mathbb{C}$. Note that the module $V_{2\pi_1+\pi_{n-1}}$ does not contain the weight space of the weight $\pi_1 + \pi_2 + 2\pi_{n-1}$. Hence, there exist $\xi_1 \in V_{3\pi_1+2\pi_{n-1}}$ and $\xi_2 \in V_{\pi_1+\pi_2+2\pi_{n-1}}$ such that $\xi = \xi_1 + \xi_2$. We obtain that $\xi_2 \neq 0$ and $\xi_2 \notin \mathcal{R}^\nabla(\mathfrak{u}(p,q)) \otimes \mathbb{C}$ this shows that $V_{\pi_1+\pi_2+2\pi_{n-1}} \not\subset \mathcal{R}^\nabla(\mathfrak{u}(p,q)) \otimes \mathbb{C}$. Similarly, $V_{2\pi_1+\pi_{n-2}+\pi_{n-1}} \not\subset \mathcal{R}^\nabla(\mathfrak{u}(p,q)) \otimes \mathbb{C}$.

The submodule $2V_{2\pi_1+\pi_{n-1}} \oplus 2V_{\pi_1+2\pi_{n-1}} \subset (\mathbb{R}^{2p,2q} \otimes \mathbb{C}) \otimes (\mathcal{R}(\mathfrak{u}(p,q)) \otimes \mathbb{C})$ defines two irreducible submodules in $\mathbb{R}^{2p,2q} \otimes \mathcal{R}(\mathfrak{u}(p,q))$ isomorphic to the irreducible $\mathfrak{u}(p,q)$-module $V_{\pi_1+2\pi_{n-1}}$. Similarly, the submodule $2V_{\pi_1} \oplus 2V_{\pi_{n-1}} \subset (\mathbb{R}^{2p,2q} \otimes \mathbb{C}) \otimes (\mathcal{R}(\mathfrak{u}(p,q)) \otimes \mathbb{C})$ defines two irreducible submodules in $\mathbb{R}^{2p,2q} \otimes \mathcal{R}(\mathfrak{u}(p,q))$ isomorphic to $\mathbb{R}^{2p,2q}$. We will show that $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$ contains only one copy of each of these submodules, we denote them by $\mathcal{Q}'$ and $\mathcal{Q}_1$, respectively. For this we turn to the space $\mathcal{P}(\mathfrak{u}(p,q))$. From the results of Leistner [11], it follows that

$$\mathcal{P}(\mathfrak{u}(p,q)) \otimes \mathbb{C} \simeq (\mathfrak{gl}(n,\mathbb{C}) \subset V)^{(1)} \oplus (\mathfrak{gl}(n,\mathbb{C}) \subset \bar{V})^{(1)},$$

where $(\mathfrak{gl}(n,\mathbb{C}) \subset V)^{(1)}$ denotes the first prolongation. It holds,

$$(\mathfrak{gl}(n,\mathbb{C}) \subset V)^{(1)} \simeq \odot^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n \simeq V_{\pi_1+2\pi_{n-1}} \oplus V_{\pi_{n-1}}.$$

Thus,

$$\mathcal{P}(\mathfrak{u}(p,q)) \simeq \odot^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n.$$

This isomorphism has the following explicate form given in [8]. Let $S \in \odot^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n \subset (\mathbb{C}^n)^* \otimes \mathfrak{gl}(n,\mathbb{C})$. We fix an identification $\mathbb{C}^n = \mathbb{R}^{2p,2q} = \mathbb{R}^{p,q} \oplus i\mathbb{R}^{p,q}$ and choose an orthonormal basis $e_1, ..., e_n$ of $\mathbb{R}^{p,q}$. Define the complex numbers S_{abc}, $a, b, c = 1, ..., n$ such that $S(e_a)e_b = \sum_c S_{acb} e_c$. It holds $S_{abc} = S_{cba}$. Define a map $S_1 : \mathbb{R}^{2p,2q} \to \mathfrak{gl}(2n,\mathbb{R})$ by the conditions $S_1(e_a)e_b = \sum_c \overline{S_{abc}} e_c$, $S_1(ie_a) = -iS_1(e_a)$, and $S_1(e_a)ie_b = iS_1(e_a)e_b$. The map $P = S - S_1 : \mathbb{R}^{2n} \to \mathfrak{gl}(2n,\mathbb{R})$ belongs to $\mathcal{P}(\mathfrak{u}(p,q))$ and any element of $\mathcal{P}(\mathfrak{u}(p,q))$ is of this form. Such element belongs to $\mathcal{P}_0(\mathfrak{u}(p,q))$ if and only if $\sum_b S_{abb} = 0$ for all $a = 1, ..., n$, i.e. $S \in (\odot^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n)_0$. Next,

$$\mathcal{P}(\mathfrak{u}(p,q)) = \mathcal{P}_0(\mathfrak{u}(p,q)) \oplus \mathcal{P}_1(\mathfrak{u}(p,q)) \simeq (\odot^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n)_0 \oplus \mathbb{R}^{p,q}.$$

According to this, any $P \in \mathcal{P}(\mathfrak{u}(p,q))$ admits the decomposition

$$P = \left(P - \frac{1}{2(n+1)} \left(\frac{\mathrm{id}}{2} \wedge_J g \right) (\widetilde{\mathrm{Ric}}(P), \cdot) \right) + \frac{1}{2(n+1)} \left(\frac{\mathrm{id}}{2} \wedge_J g \right) (\widetilde{\mathrm{Ric}}(P), \cdot). \tag{65}$$

Any linear map $Q:\mathbb{R}^{2p,2q}\to\mathcal{R}(\mathfrak{u}(p,q))$ defines the map

$$P:\mathbb{R}^{2p,2q}\to\mathfrak{u}(p,q),\quad P(X)=(\mathrm{tr}_{(1,5)})Q.(\cdot,\cdot,X,\cdot)=g^{ab}Q_{X_a}(X,X_b). \tag{66}$$

If $Q\in\mathcal{R}^\nabla(\mathfrak{u}(p,q))$, then the element $\mathrm{tr}_{(1,5)}\,Q$ belongs to $\mathcal{P}(\mathfrak{u}(p,q))$. For any $P\in\mathcal{P}(\mathfrak{u}(p,q))$ define the tensors $\Psi_1(P)$ and $\Psi_2(P)$ as follows:

$$\Psi_1(P)_X=JP(JX)\wedge_J g, \tag{67}$$

$$\Psi_2(P)_X(Y,Z)=P\big((Y\wedge_J Z)X\big)+X\wedge_J(P(Z)Y-P(Y)Z). \tag{68}$$

It can be checked that $\Psi_1(P)$ and $\Psi_2(P)$ map $\mathbb{R}^{2p,2q}$ to $\mathcal{R}(\mathfrak{u}(p,q))$, and the maps Ψ_1 and Ψ_2 are $\mathfrak{u}(p,q)$-equivariant. It can be shown that if $P\neq 0$, then $\Psi_1(P)\notin\mathcal{R}^\nabla(\mathfrak{u}(p,q))$. Moreover, $(\Psi_1-\Psi_2)(P)\in\mathcal{R}^\nabla(\mathfrak{u}(p,q))$ for any $P\in\mathcal{P}(\mathfrak{u}(p,q))$. Next,

$$\mathrm{tr}_{(1,5)}((\Psi_1-\Psi_2)P_0)=2(n+3)P_0,\quad \mathrm{tr}_{(1,5)}((\Psi_1-\Psi_2)P_1)=4(n+2)P_1, \tag{69}$$

where $P_0\in\mathcal{P}_0(\mathfrak{u}(p,q))$ and $P_1\in\mathcal{P}_1(\mathfrak{u}(p,q))$. This shows that $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$ contains exactly one submodule isomorphic to $(\odot^2(\mathbb{C}^n)^*\otimes\mathbb{C}^n)_0$ and exactly one submodule isomorphic to $\mathbb{R}^{2p,2q}$.

We are left with the submodules $V_{\pi_2+\pi_{n-1}}$ and $V_{\pi_1+\pi_{n-2}}$ of $(\mathbb{R}^{2p,2q}\otimes\mathbb{C})\otimes(\mathcal{R}(\mathfrak{u}(p,q))\otimes\mathbb{C})$. These submodules define an irreducible submodule in $\mathbb{R}^{2p,2q}\otimes\mathcal{R}(\mathfrak{u}(p,q))$. This submodule is contained in $\mathbb{R}^{2p,2q}\otimes\mathcal{R}'(\mathfrak{u}(p,q))$. Hence any element of this submodule is of the form

$$\xi_X=\Upsilon_X\wedge_J g,$$

where for each X, Υ_X is symmetric and commutes with J. Clearly, the contraction of any two indexes of Υ gives 0. Suppose that $\xi\in\mathcal{R}^\nabla(\mathfrak{u}(p,q))$. Then,

$$g^{ab}g((\xi_X(X_a,Z)+\xi_{X_a}(Z,X)+\xi_Z(X,X_a))X_b,V)=0.$$

This implies

$$(2n+3)\Upsilon_X(Z,V)-(2n+3)\Upsilon_Z(X,V)-\Upsilon_{JX}(JZ,V)+\Upsilon_{JZ}(JX,V)-2\Upsilon_{JV}(JZ,X)=0.$$

Simple manipulations show

$$\Upsilon_V(X,Z)=\Upsilon_{JV}(X,JZ).$$

Finally,

$$\Upsilon_V(X,Z)=\Upsilon_{JV}(X,JZ)=-\Upsilon_{JV}(JX,Z)=-\Upsilon_V(X,Z),$$

and $\Upsilon=0$. Thus the submodule under the consideration is not contained in $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$. The theorem is proved. □

9 Proof of Theorem 4

Let (M,g,J) be a pseudo-Kählerian manifold. From Theorem 3 it follows that

$$\nabla R=Q_0+Q'+Q_1$$

for some tensors Q_0, Q', Q_1 such that the values of these tensors at any $x \in M$ belong to the submodules of $\mathcal{R}^\nabla(\mathfrak{u}(T_xM))$ isomorphic respectively to $V_{3\pi_1+2\pi_{n-1}}$, $V_{\pi_1+2\pi_{n-1}}$ and $\mathbb{R}^{p,q}$.

From (35) it follows that

$$\nabla_X R = \nabla_X B + \nabla_X K \wedge_J g.$$

Let $x \in M$. Then for each $X \in T_xM$, $\nabla_X B_x$ belongs to $V_{2\pi_1+2\pi_{n-1}} \subset \mathcal{R}(\mathfrak{u}(p,q))$, while $\nabla_X K_x \wedge_J g_x$ belongs to $\mathfrak{u}(p,q) \subset \mathcal{R}(\mathfrak{u}(p,q))$. We get

$$\nabla B_x \in \mathbb{R}^{2p,2q} \otimes V_{2\pi_1+2\pi_{n-1}} = V_{3\pi_1+2\pi_{n-1}} \oplus V_{\pi_1+\pi_2+2\pi_{n-1}} \oplus V_{\pi_1+2\pi_{n-1}}, \tag{70}$$

$$\nabla K_x \in \mathbb{R}^{2p,2q} \otimes \mathfrak{u}(p,q) = (V_{\pi_1+2\pi_{n-1}} \oplus V_{\pi_2+\pi_{n-1}} \oplus \mathbb{R}^{2p,2q}) \oplus \mathbb{R}^{2p,2q}. \tag{71}$$

Since $\mathcal{R}^\nabla(\mathfrak{u}(p,q))$ does not contain the submodule $V_{\pi_1+\pi_2+2\pi_{n-1}}$, we obtain

$$\nabla W_x \in V_{3\pi_1+2\pi_{n-1}} \oplus V_{\pi_1+2\pi_{n-1}}. \tag{72}$$

Since

$$g^{ab}\nabla_{X_a} R_x(\cdot, X_b) \in \mathcal{P}(\mathfrak{u}(p,q)),$$

this element can be decomposed in the form

$$g^{ab}\nabla_{X_a} R_x(\cdot, X_b) = -\tilde{D}_x + \left(\frac{\mathrm{id}}{2} \wedge_J g\right)(V, \cdot) \tag{73}$$

for some $\tilde{D}_x \in \mathcal{P}_0(\mathfrak{u}(p,q))$ and $V \in \mathbb{R}^{2p,2q}$. From the above it follows that there are unique elements $P_{0x} \in \mathcal{P}_0(\mathfrak{u}(p,q))$ and $P_{1x} \in \mathcal{P}_1(\mathfrak{u}(p,q))$ such that

$$Q_{1x} = (\Psi_1 - \Psi_2)P_{0x}, \quad Q'_x = (\Psi_1 - \Psi_2)P_{1x}.$$

Let us find these elements. It holds

$$\mathrm{tr}_{(1,5)}(\nabla R_x) = \mathrm{tr}_{(1,5)}\, Q_{1x} + \mathrm{tr}_{(1,5)}\, Q'_x = 2(n+3)P_{0x} + 4(n+2)P_{1x}.$$

We conclude that

$$P_{0x} = -\frac{1}{2(n+3)}\tilde{D}_x, \quad P_{1x} = \frac{1}{4(n+2)}\left(\frac{\mathrm{id}}{2} \wedge_J g\right)(V, \cdot). \tag{74}$$

The last equality implies

$$\widetilde{\mathrm{Ric}}(P_{1x}) = -\frac{n+1}{2(n+2)}V.$$

Taking a trace in (73), we get

$$\mathrm{grad}\, s = 4(n+1)V.$$

Summarizing all the above, we obtain that

$$P_1 = \frac{1}{16(n+2)(n+1)}\left(\frac{\mathrm{id}}{2} \wedge_J g\right)(\mathrm{grad}\, s, \cdot),$$

and $\tilde{D}$ satisfies (38), (39), (37). Thus we have found Q_1 and Q'. The component Q_0 can be found as

$$Q_0 = \nabla R - Q_1 - Q'.$$

Theorem 4 is proved. $\square$

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